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An amazing optimisation problem

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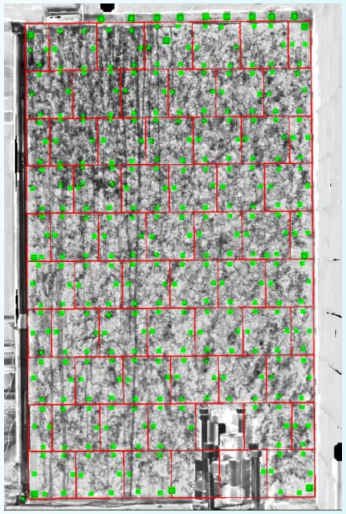

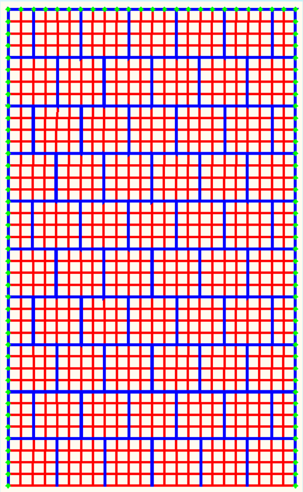

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An amazing optimisation problem

OUTLINE:

- Background: some results from photogrammetry
- The (VERY) simplified optimisation problem
- EXCE, steepest descent, conjugate gradient & BFGS
- The Gauss-Newton method (+ new operator GANE)
- Conclusions, further works & bibliography

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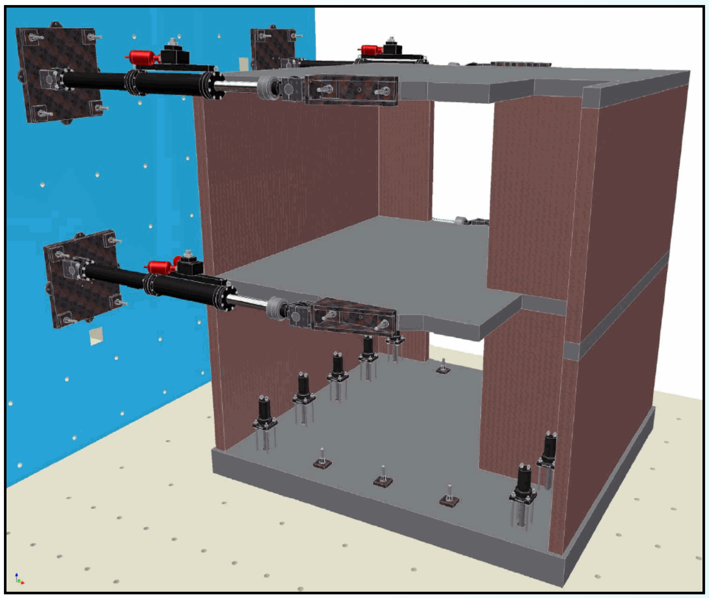
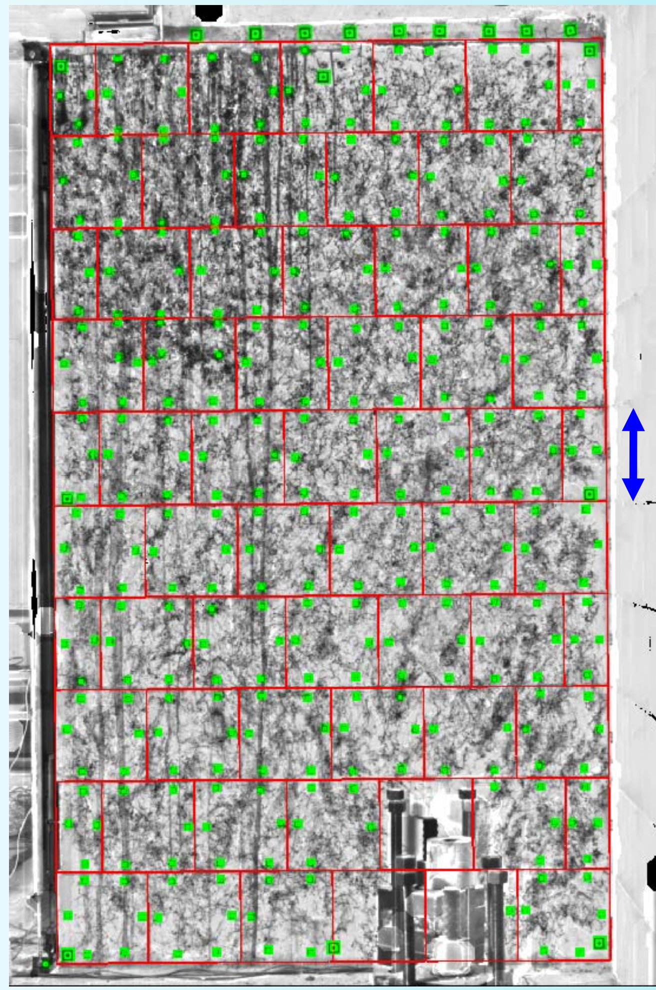
Background: some results from photogrammetry

The structure ESECMASE



The wall



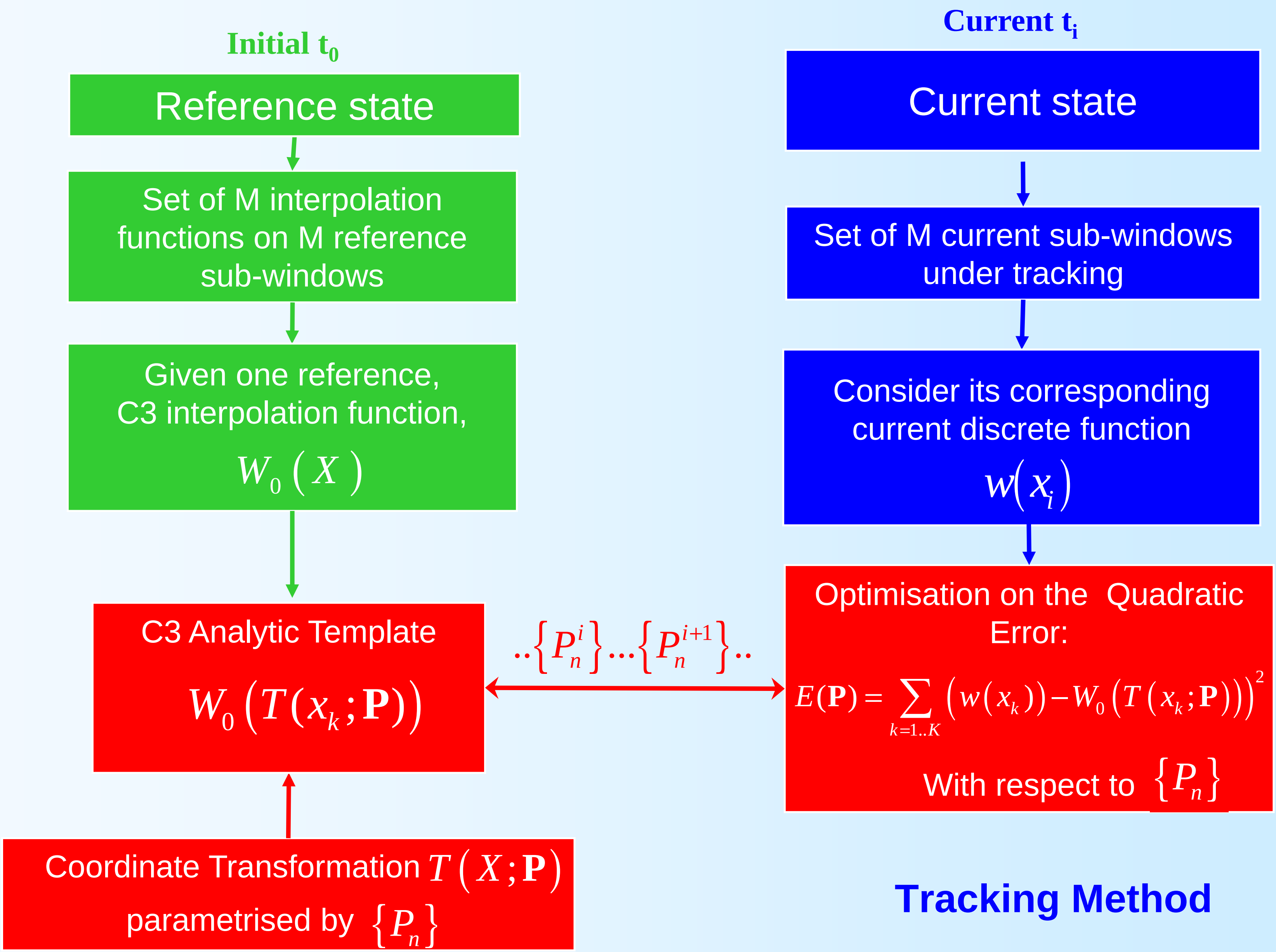
20cm

The camera

4096 grey levels, S/N 64 dB,
1536x1024 pixels
Kodak CCD KAF 1602E

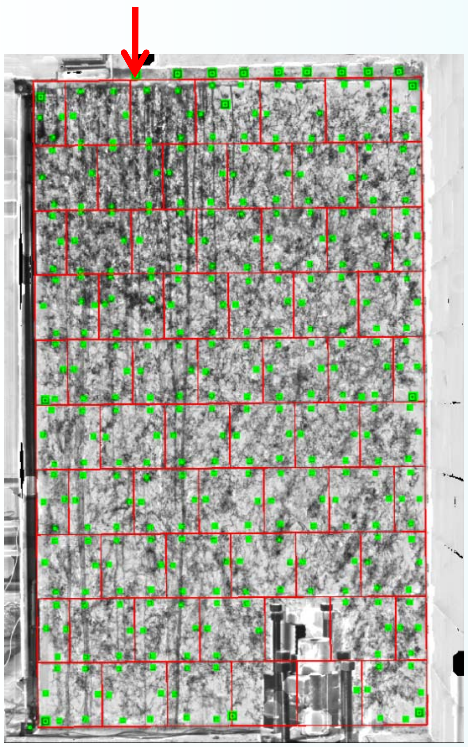
+-1.3mm/pixel

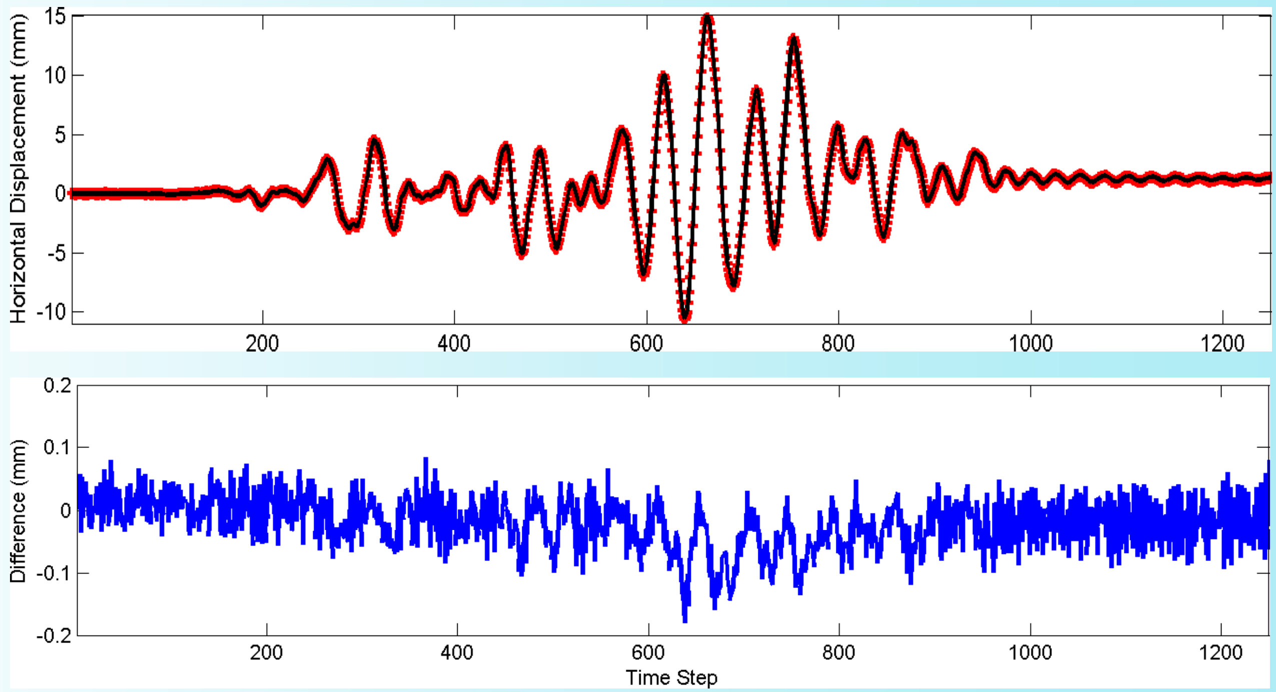
Background: some results from photogrammetry



Background: some results from photogrammetry

Comparison with Heidenhain measurement (K14 - 0.16g)





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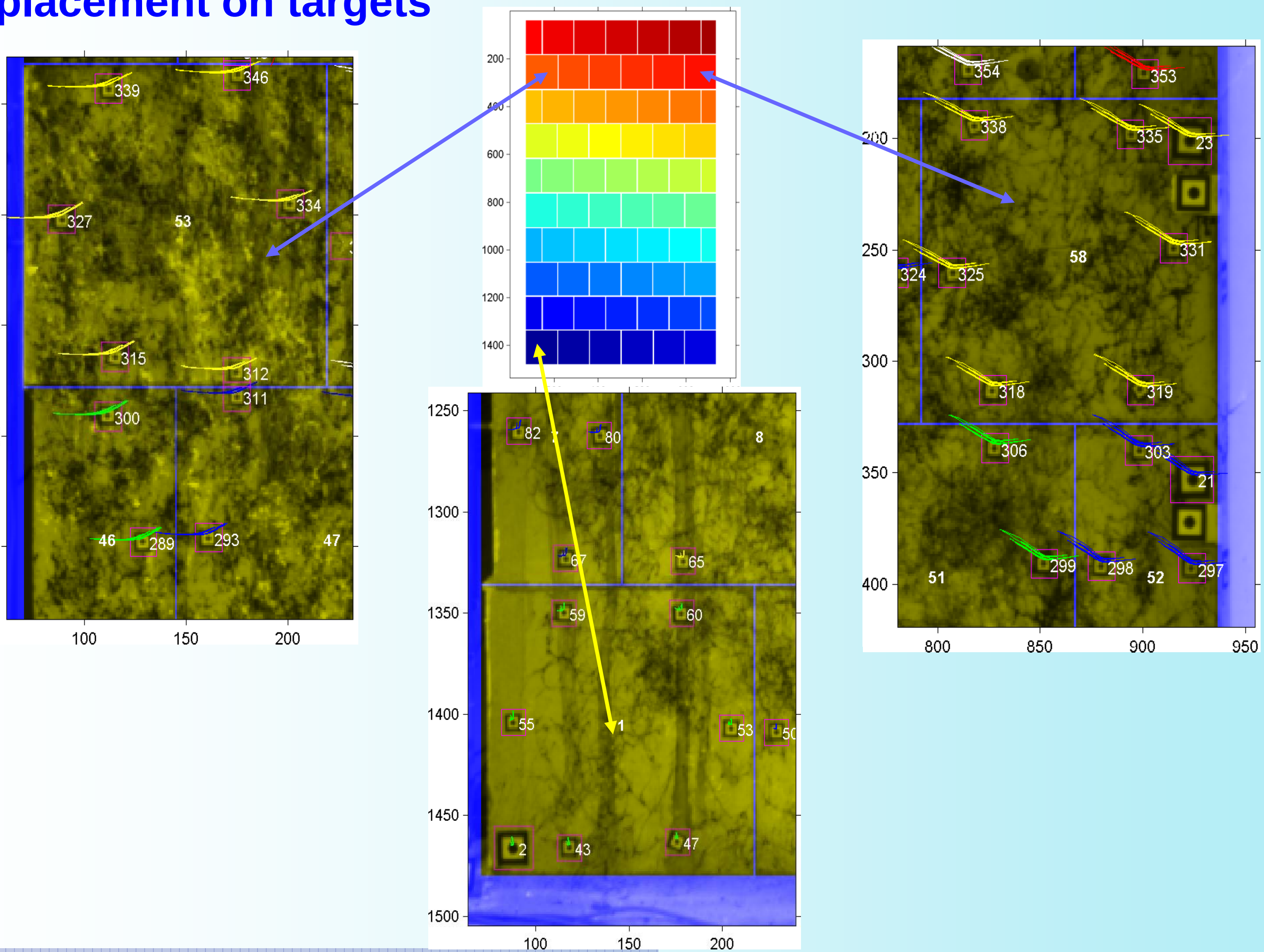
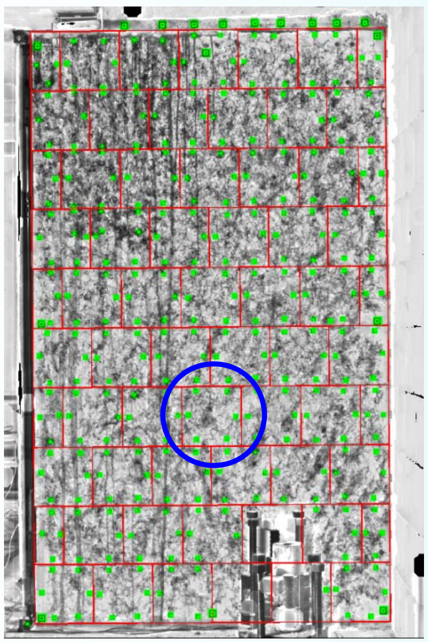
Background: some results from photogrammetry

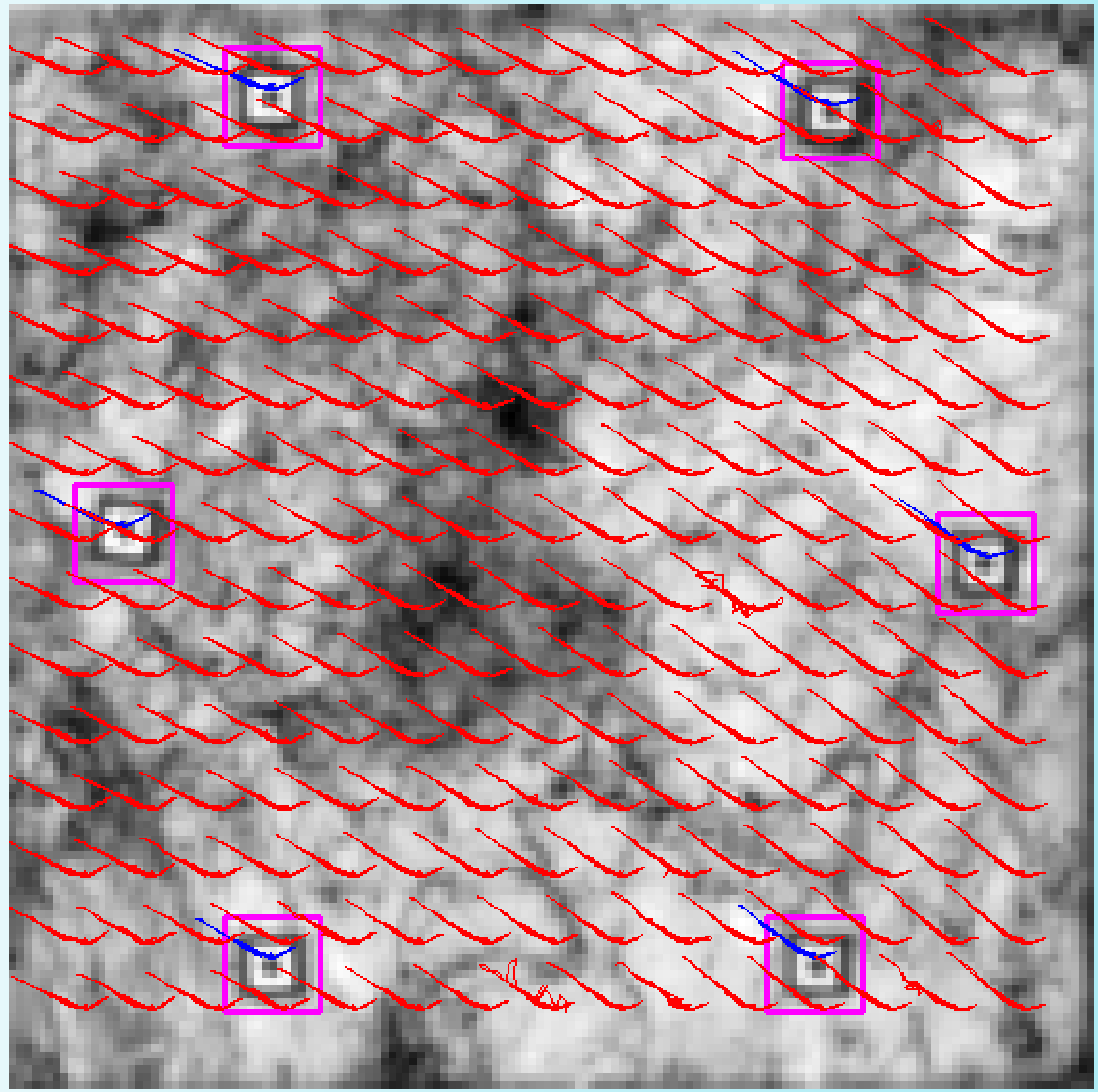
Displacement on targets



Background: some results from photogrammetry

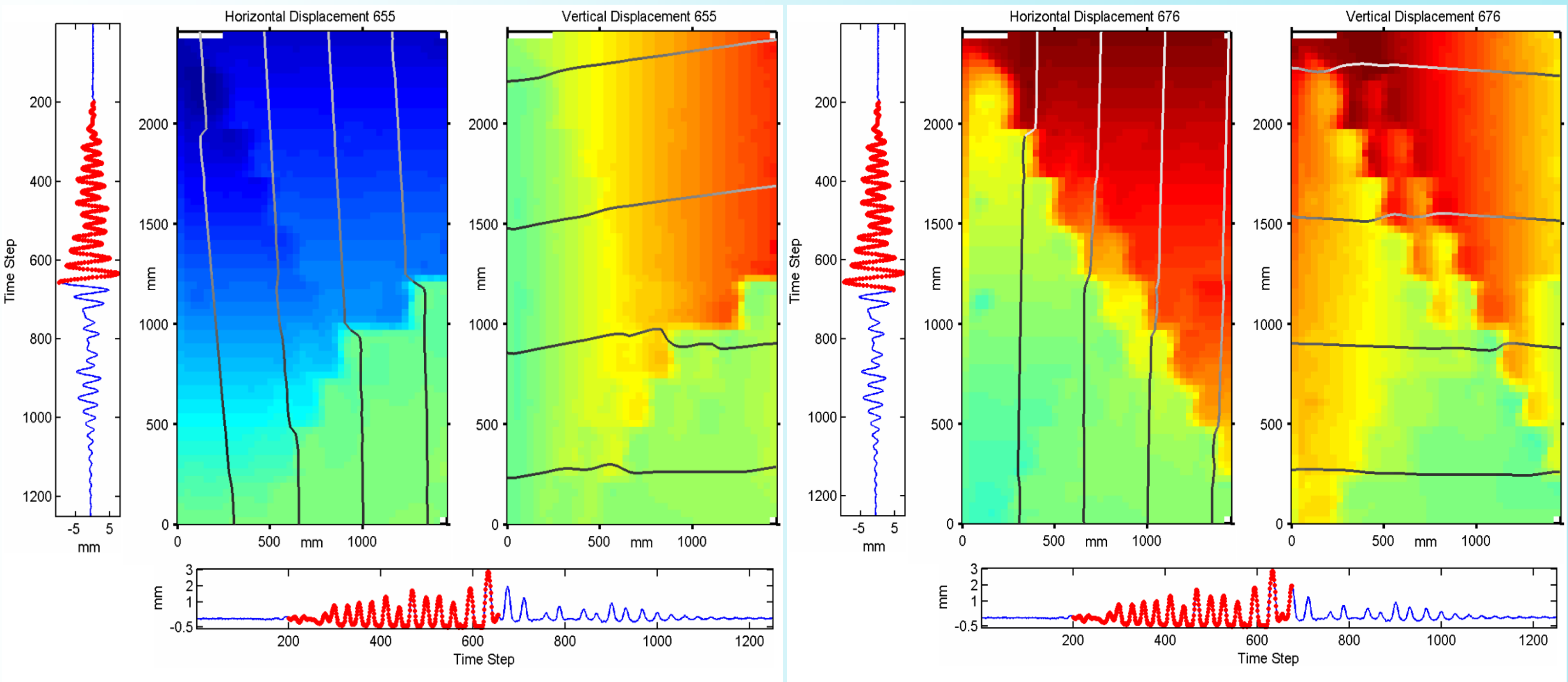
Displacement on texture





Background: some results from photogrammetry

Displacement fields (K12 - 0.12g)



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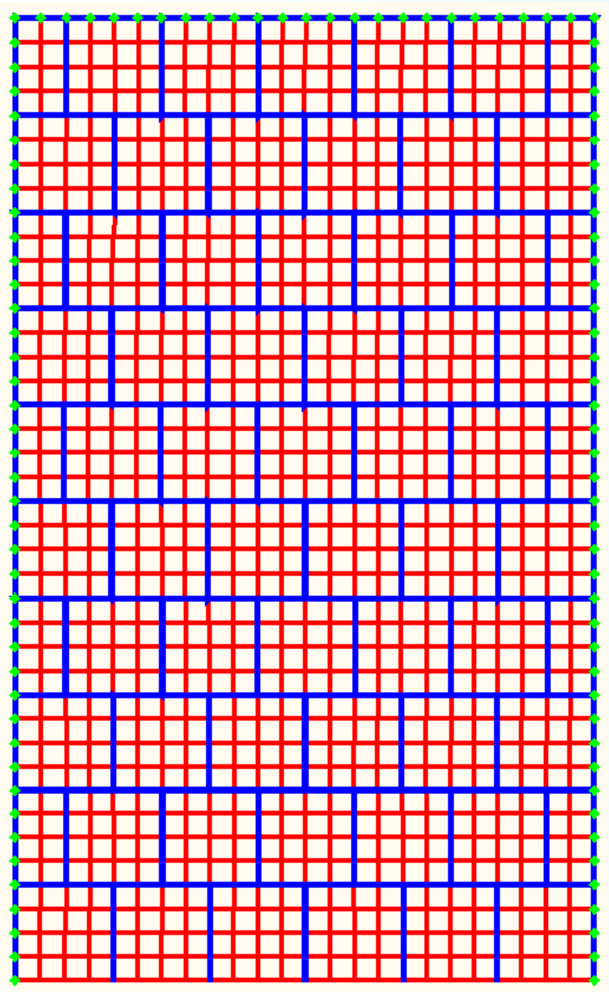
The (VERY) simplified optimisation problem



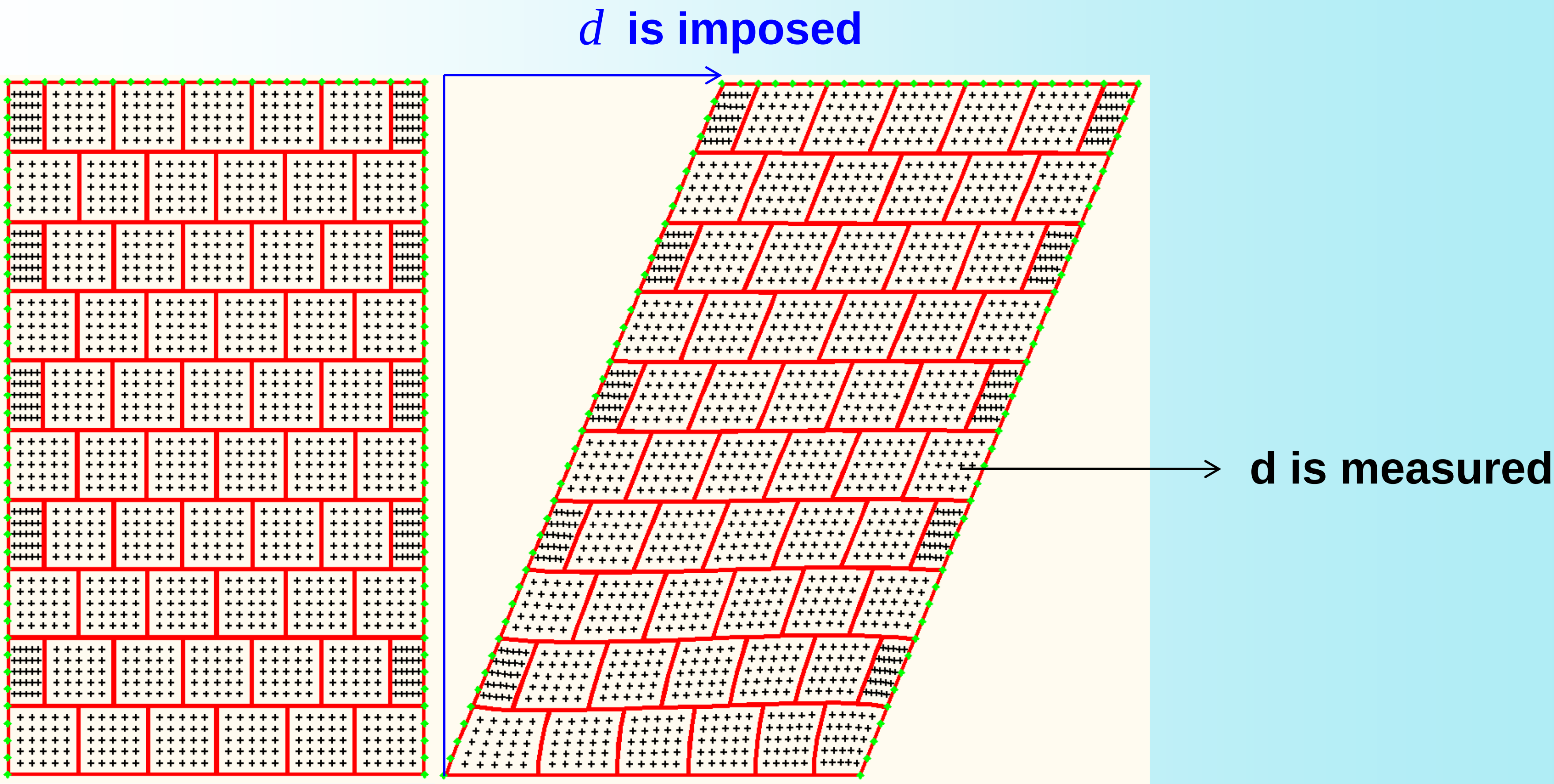
In the elastic case…

65 bricks → $E_i, i = 1, \ldots, 65 \qquad \nu = 0.3$

Joints → $E_{66} \rightarrow K_n = \frac{E_{66}}{e}, K_s = \frac{E_{66}}{2e(1+\nu)}$

Identification of $\mathbf{E}$ **???**

The (VERY) simplified optimisation problem



The reference solution $\mathbf{d}^{+}(\mathbf{E}^{+})$ is obtained with $\overline{\mathbf{E}}^{+} = 1$ and $\sigma(\mathbf{E}^{+}) = 5\%$

The (VERY) simplified optimisation problem

Find $$\mathbf{E}^* = \operatorname{argmin}_{\mathbf{E}} \{ F(\mathbf{E}) \}$$

Where $$F(\mathbf{E}) = \left\| \mathbf{f}(\mathbf{E}) \right\| = \sqrt{\mathbf{f}(\mathbf{E})^{\mathrm{T}} \mathbf{f}(\mathbf{E})}$$

$$\mathbf{f}(\mathbf{E}) = \mathbf{d}(\mathbf{E}) - \mathbf{d}^+(\mathbf{E}^+)$$

(first) problem $\mathbf{d}(\mathrm{E}) = \mathbf{d}(\lambda \mathrm{E})$ **!!!**

The (VERY) simplified optimisation problem

Numerical derivation of the gradients

$$\frac{\partial F(\mathrm{E})}{\partial E_i} \approx \frac{F(E_1,\ldots,E_i+\delta E,\ldots,E_n)-F(E_1,\ldots,E_i-\delta E,\ldots,E_n)}{2\delta E}$$

And also (later)

$$\frac{\partial \mathbf{f}(\mathrm{E})}{\partial E_i} \approx \frac{\mathbf{f}(E_1,\ldots,E_i+\delta E,\ldots,E_n)-\mathbf{f}(E_1,\ldots,E_i-\delta E,\ldots,E_n)}{2\delta E}$$

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EXCE, steepest descent, conjugate gradient & BFGS

EXCE

```
The EXCELL operator computes the minimum of a function F(Xi),the use
method is the well-known MMA (Method of Moving Asymptotes) proposed
by K.Svanberg. The purpose is to find the minimum of a function F(Xi)
with i=1,N given that :

   - the relations Cj(Xi) < Cjmax (j>0 and j=1,M) must be satisfied

   - there are relations on each unknown  Ximin <  Xi < Ximax

    The functions F and Cj are defined by the function values and
 by their derivatives at the starting point X0.
```

$F(E_i), C_j(E_i)$ **?????**

EXCE, steepest descent, conjugate gradient & BFGS

EXCE

$$F(\mathbf{E}) = \left\| \mathbf{d}(\mathbf{E}) - \mathbf{d}^{+}(\mathbf{E}^{+}) \right\|$$

1st case

$$C_1(\mathbf{E}) = \sigma(\mathbf{E}) - \sigma(\mathbf{E}^{+}) < 0$$

$$C_2(\mathbf{E}) = \overline{\mathbf{E}} - \overline{\mathbf{E}}^{+} < 0 \ , \ C_3(\mathbf{E}) = \overline{\mathbf{E}}^{+} - \overline{\mathbf{E}} < 0$$

2nd case

$$F(\mathbf{E}) = \left\| \mathbf{d}(\mathbf{E}) - \mathbf{d}^{+}(\mathbf{E}^{+}) \right\|$$

$$C_1(\mathbf{E}) = \sigma(\mathbf{E}) - \sigma(\mathbf{E}^{+}) < 0 \ , \ C_2(\mathbf{E}) = (\overline{\mathbf{E}} - \overline{\mathbf{E}}^{+})^2 < (\overline{\mathbf{E}}^{+} / 100) \ \ or \ 0$$

3rd case

$$F(\mathbf{E}) = \sigma(\mathbf{E})$$

$$C_1(\mathbf{E}) = \left\| \mathbf{d}(\mathbf{E}) - \mathbf{d}^{+}(\mathbf{E}^{+}) \right\| / \left\| \mathbf{d}^{+}(\mathbf{E}^{+}) \right\| < 1.0 \times 10^{-4}$$

… but no convincing results

EXCE, steepest descent, conjugate gradient & BFGS

Descent method: general format

Descent direction $\mathbf{E}_d^i$ **with** $\mathbf{E}_d^{i\,T} \nabla F\big|_{\mathbf{E}^i} < 0$

Line search $\alpha^i = \mathrm{argmin}_{\alpha>0} \left\{ F(\mathbf{E}^i + \alpha \mathbf{E}_d^i) \right\}$ **Robust line search???**

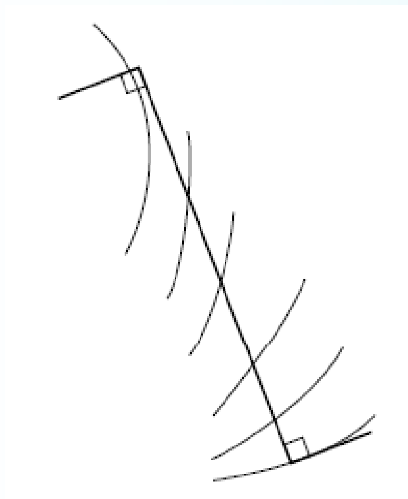
$\mathbf{E}^{i+1} = \mathbf{E}^i + \alpha \mathbf{E}_d^i$ **and loop on** i **until convergence**

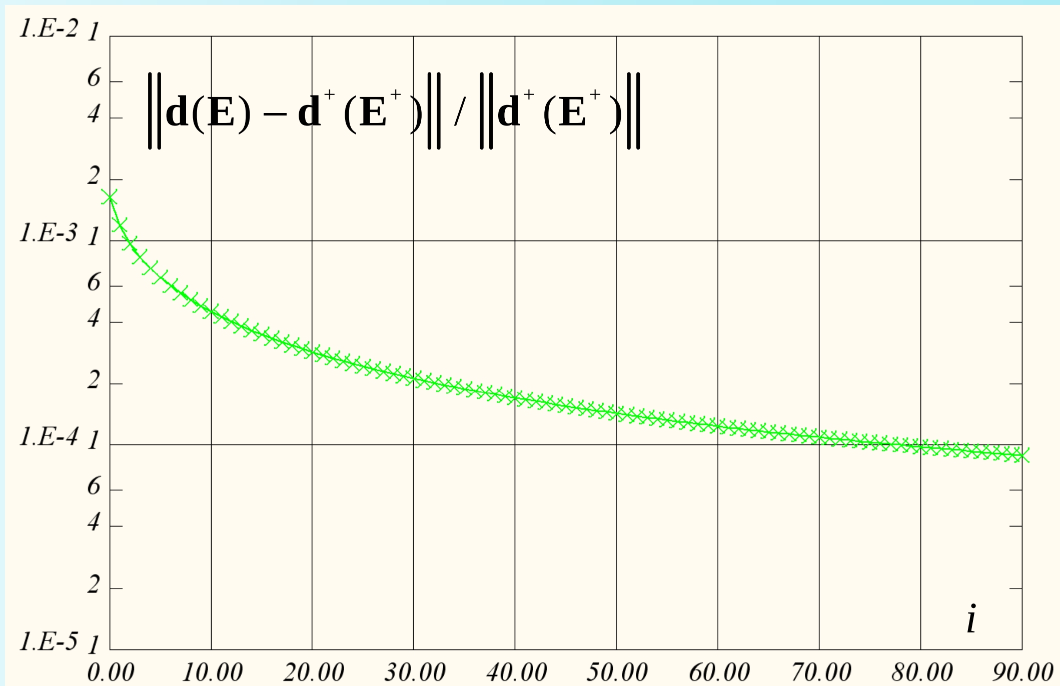
Meaning of convergence???

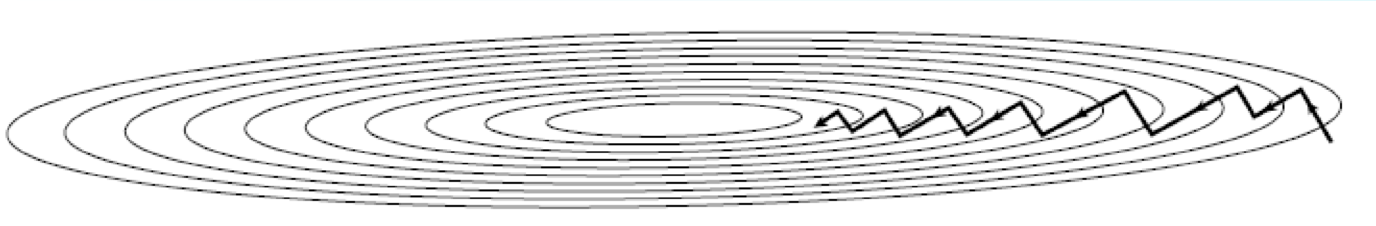
Here we concentrate on $\mathbf{E}_d^i$ **!!!!!!!**

EXCE, steepest descent, conjugate gradient & BFGS

Steepest descent $\mathbf{E}_{st}^{i} = -\nabla F\big|_{\mathbf{E}^{i}}$







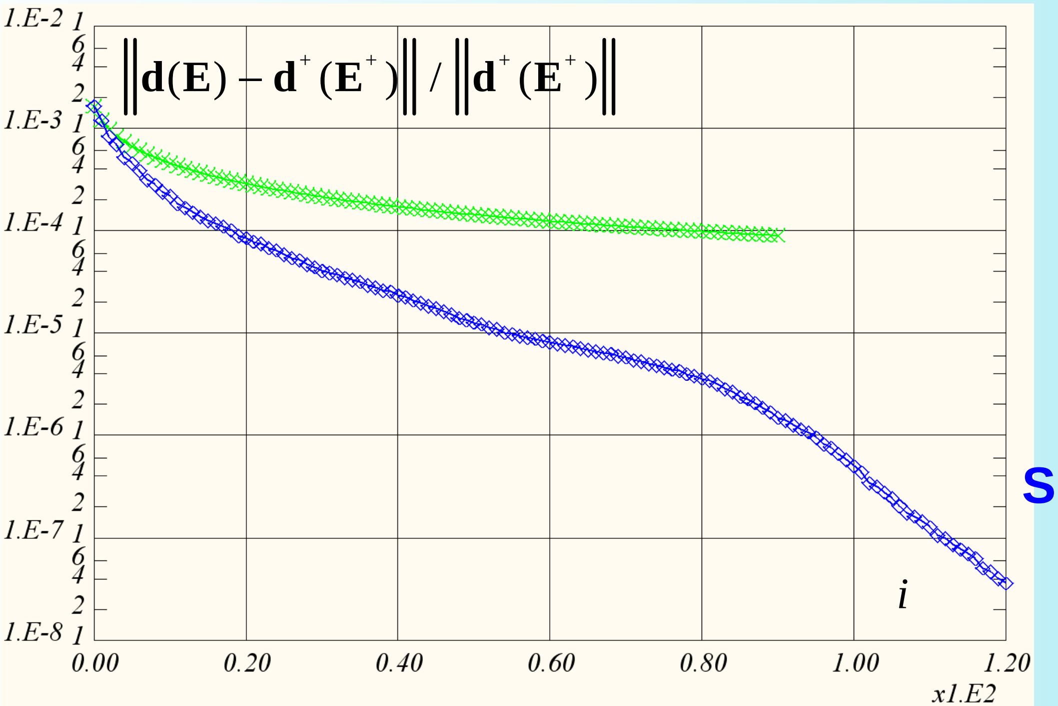
The convergence can be very slow!

EXCE, steepest descent, conjugate gradient & BFGS

Conjugate gradient : the descent directions are mutually orthogonal

$$\mathbf{E}^i_{gc} = \mathbf{g}^i + \gamma^i \mathbf{E}^{i-1}_{gc} \quad \mathbf{g}^i = -\nabla F\big|_{\mathbf{E}^i} \quad \gamma^i = \frac{(\mathbf{g}^i - \mathbf{g}^{i-1})^T \mathbf{g}^i}{\left\|\mathbf{g}^{i-1}\right\|^2}$$

(with precise line search)

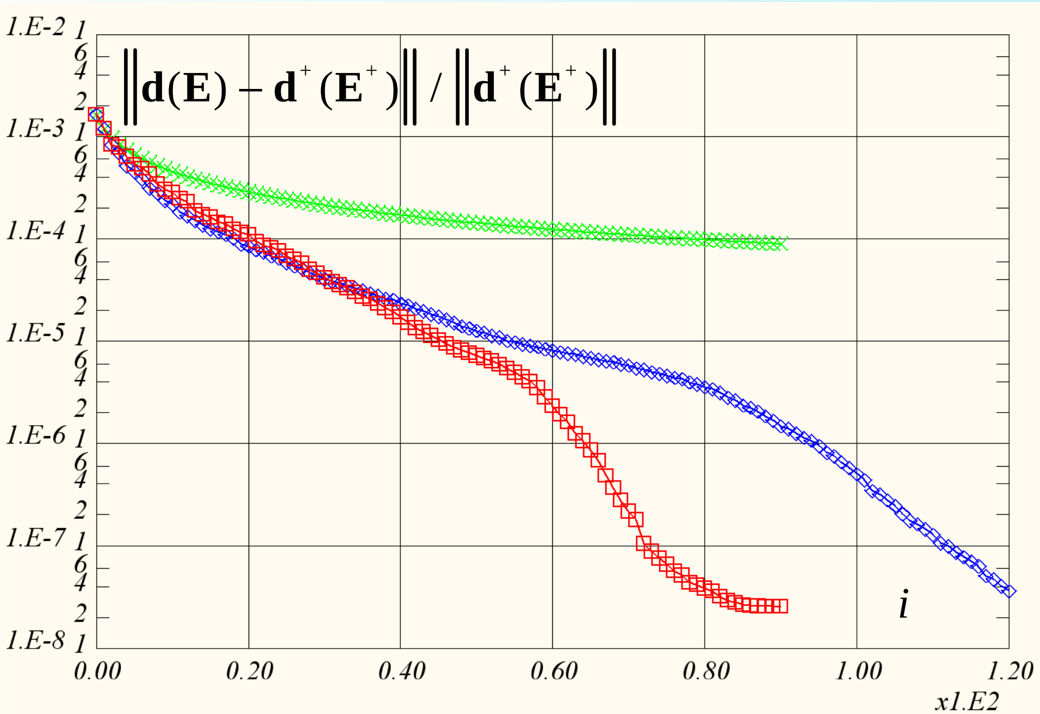


Surprisingly slow convergence

EXCE, steepest descent, conjugate gradient & BFGS

BFGS: the Hessian matrix is iteratively constructed

$$\mathbf{E}^{i}_{bfgs} = -\mathbf{H}^{i}_{bfgs} \nabla F \Big|_{\mathbf{E}^i} \qquad \mathbf{H}^{i}_{bfgs} = \mathbf{H}^{i-1}_{bfgs} + \ldots$$



BFGS=Broyden-Fletcher-Goldfarb-Shanno

Surprisingly slow convergence

EXCE, steepest descent, conjugate gradient & BFGS

Summary:

- **Very slow convergence**
- **"High" level asymptotic convergence plateau**

Conclusion: we need to do better!

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The Gauss-Newton method (+ new operator GANE)

Write the problem as a least square problem…

Find $$\mathbf{E}^* = \operatorname{argmin}_{\mathbf{x}} \{F(\mathbf{E})\}$$

Where $$F(\mathbf{E}) = \frac{1}{2}\sum_{i=1}^{m}\left(f_i(\mathbf{E})\right)^2 = \frac{1}{2}\left\|\mathbf{f}(\mathbf{E})\right\|^2 = \frac{1}{2}\mathbf{f}(\mathbf{E})^{\mathrm{T}}\mathbf{f}(\mathbf{E})$$

$$\mathbf{f}(\mathbf{E}) = \mathbf{d}(\mathbf{E}) - \mathbf{d}^+(\mathbf{E}^+)$$

… and use it!

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The Gauss-Newton method (+ new operator GANE)

Same cost as computing $\partial F / \partial E_j$

The Gauss-Newton Method

Using $\mathbf{f}(\mathbf{E}+\delta\mathbf{E}) \simeq \mathbf{f}(\mathbf{E}) + \mathbf{J}(\mathbf{E})\delta\mathbf{E}$ with $\left[\mathbf{J}(\mathbf{E})\right]_{ij} = \frac{\partial f_i}{\partial E_j}(\mathbf{E})$

$\mathbf{E}_{gn} = \mathrm{argmin}_{\delta\mathbf{E}}\left\{F(\mathbf{E}+\delta\mathbf{E})\right\}$ is given by $(\mathbf{J}^T\mathbf{J})\mathbf{E}_{gn} = -\mathbf{J}^T\mathbf{f} = -\nabla F$

→ Use $\mathbf{E}_{gn}$ as a descent direction

Warning: no indetermination if the columns of $\mathbf{J}$ are linearly independent

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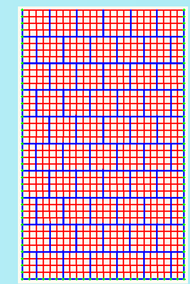
The Gauss-Newton method (+ new operator GANE)

Operator GA(us-)NE(wton

```
TAB2=GANE TAB1;

CHPO1 RIGI1=GANE TAB1 'MATR';

Description:
____________

The function to minimize is 2F(X)=f(X).f(X), and J(X)=df/dX

The descent direction is H solution of:
  transpose(J)J  H  = -transpose(J)f

Contents:
_________

TAB1    :  Table of type 'VECTEUR' containing
           TAB1 . 0 = f(X)
           TAB1 . 1 = df/dX1
                .
           TAB1 . n = df/dXn

CHPO1   :  -transpose(J)f
RIGI1   :  transpose(J)J

TAB2    :  TABLE of type 'VECTEUR' containing
           TAB2 . 1 = H1
                .
           TAB2 . n = Hn
```

The Gauss-Newton method (+ new operator GANE)

There is a second problem!!! (first problem $\mathbf{d}(\mathbf{E})=\mathbf{d}(\lambda\mathbf{E})$)



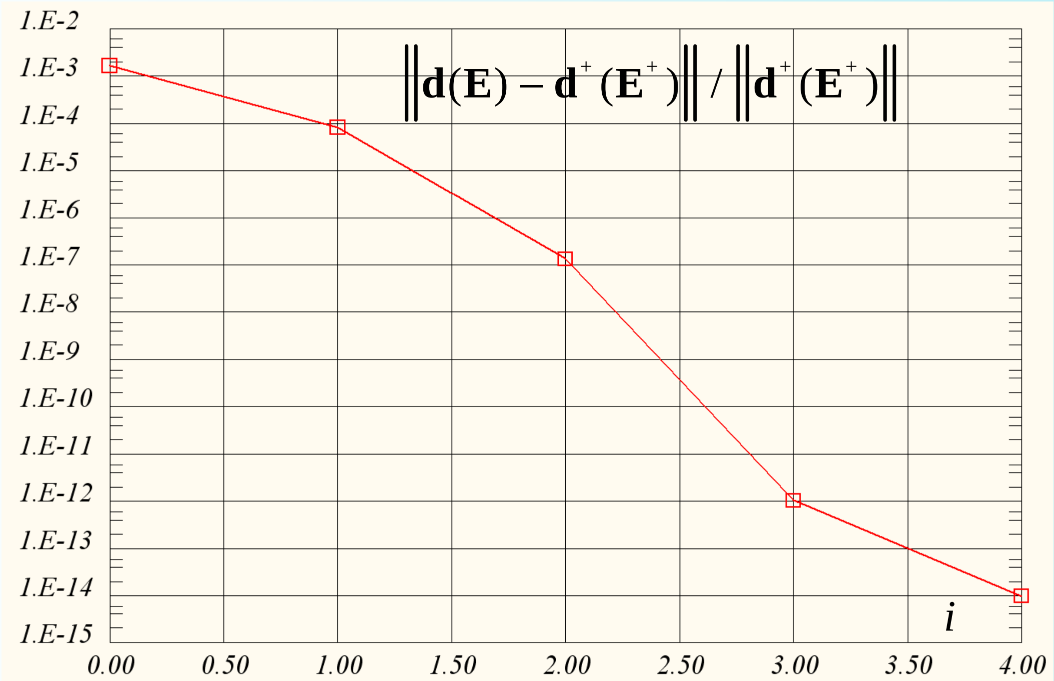
- $(\mathbf{J}^T\mathbf{J})$ is almost always rank 1 deficient (use of RESO 'ENSE')
- For instance: $\dfrac{\partial \mathbf{f}}{\partial E_{66}}=\sum_{i=1}^{65} a_i(\mathbf{E}_{66})\dfrac{\partial \mathbf{f}}{\partial E_i}$ ($a_i=-1$ for $\mathbf{E}=\{1\}$)

BUT

- GANE can be used to find out the a_i ($(\mathbf{J}_{65}^T\mathbf{J}_{65})\mathbf{a}=\mathbf{J}_{65}\dfrac{\partial \mathbf{f}}{\partial E_{66}}$)
- Assuming $E_{66}=g(\mathbf{E}_{65})$ with $\partial g/\partial E_i=a_i$ → $\tilde{\mathbf{J}}_{65}$ with $\widetilde{\dfrac{\partial \mathbf{f}}{\partial E_i}}_{65}=\dfrac{\partial \mathbf{f}}{\partial E_i}+a_i\dfrac{\partial \mathbf{f}}{\partial E_{66}}$
- Use GANE again for finding $\mathbf{E}_{gn65}$ and complete with $E_{gn66}=\sum_{i=1} a_i E_{gni}$

The Gauss-Newton method (+ new operator GANE)

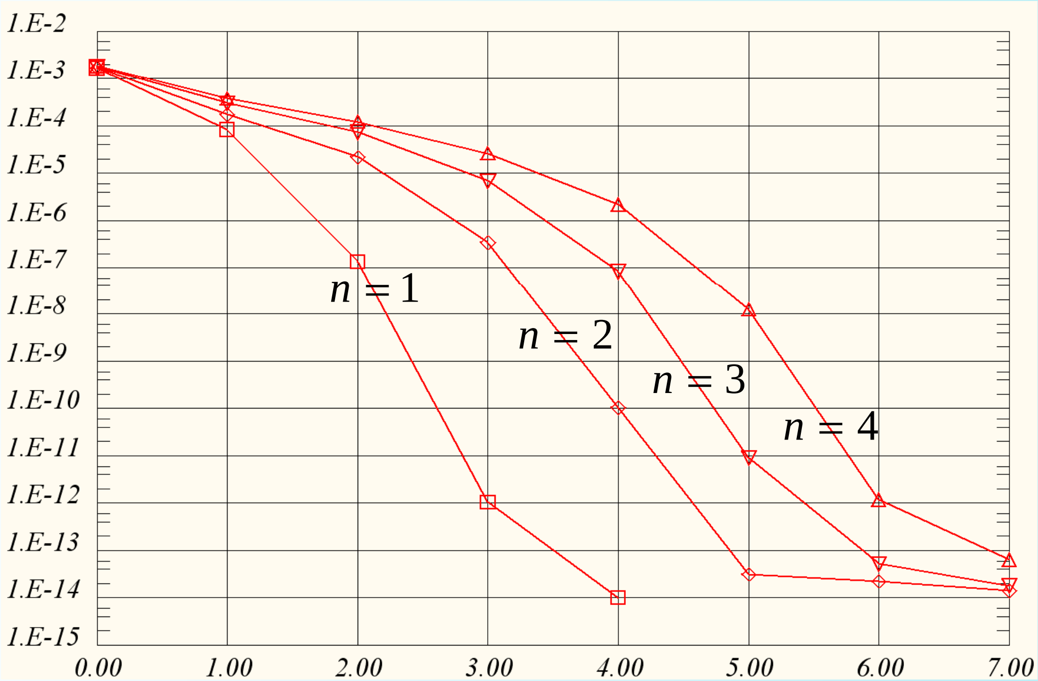
And finally!



… with $\max\left(\left|E_i - E_i^{+}\right|\right) \approx 10^{-7}$

The Gauss-Newton method (+ new operator GANE)

Substituting E_{66}^{+} **with** nE_{66}^{+}, $n = 1, 2, 4, 8$ **(and still starting from** $\mathbf{E} = \{1\}$ **)**



WARNING

- **Sometimes RESO takes care itself of the indetermination…**
- $\max\left(\left|\frac{E_i}{\overline{\mathbf{E}}} - \frac{E_i^{+}}{\overline{\mathbf{E}}^{+}}\right|\right) \approx 10^{-7}$ **with** $\overline{\mathbf{E}} \neq \overline{\mathbf{E}}^{+}$

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Conclusions, further works & bibliography

Conclusions & further works

- **Nothing is given for free!**
- **Cast3M demonstrates again its ability to help to understand (and solve) problems!**
- **Gauss-Newton is the root for other classical methods**
  - **Damped Gauss-Newton = Levenberg-Marquardt method**
    $$(\mathbf{J}^T\mathbf{J} + \mu\mathbf{I})\mathbf{E}_{lm} = -\mathbf{J}^T\mathbf{f} = -\nabla F$$ **(already in GANE)**
  - **Powell's Dog Leg Method**
- **Use of noisy experimental data**
- **Identification of non-linear models with more refined meshes**
- **Line search, convergence tests…**

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Conclusions, further works & bibliography

Short bibliography

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- H. Matthis & G. Strang, The Solution of Non-Linear Finite Element Equations IJNME, 14, 1613-1626, 1979
- K. Madsen, H.B. Nielsen & O. Tingleff Methods for Non-Linear Least Squares Problems Technical University of Denmark, 2004
- T. Charras & J. Kichenin L'optimisation dans CAST3M